# ASYMPTOTICS FOR THE ENUMERATION OF COMMUTING MATRICES OVER FINITE FIELDS

KATHRIN BRINGMANN, SHANE CHERN, JOHANN FRANKE,
AND BERNHARD HEIM

*In honor of James Lepowsky's 80-th birthday.*

ABSTRACT. We give asymptotic expressions for the number of commuting matrices over finite fields. For this we use product expansions for the corresponding generating functions.

## 1. INTRODUCTION AND STATEMENT OF RESULTS

Let $\mathbb{F}_{p^r}$ be the finite field with $p^r$ elements, where $p$ is a prime and $r \in \mathbb{N}$. The study of the enumeration of commuting square matrix pairs over $\mathbb{F}_{p^r}$ was first launched by Feit and Fine [5]. Let $Q_{p^r}(n)$ denote the number of ordered pairs of (not necessarily distinct) commuting $n \times n$ matrices with coefficients in $\mathbb{F}_{p^r}$. Feit and Fine proved that $Q_{p^r}(n)$ (re-normalized) admits the following product-like generating function.

**Theorem 1.1** (Feit–Fine, [5, p. 91]). *We have*

$$\sum_{n \geq 0} \frac{Q_{p^r}(n)}{p^{rn^2} f_{p^r}(n)} w^n = \prod_{\substack{\ell \geq 1 \\ j \geq 0}} \frac{1}{1 - p^{r(1-j)} w^\ell}, \tag{1.1}$$

*where*

$$f_{p^r}(n) := \prod_{j=1}^{n} \left(1 - p^{-rj}\right). \tag{1.2}$$

**Remarks.**

(1) *The product in (1.1) converges for $w \in \mathbb{C}$ with $|w| < 1$ not being a pole and continues to a meromorphic function for $|w| < 1$. This is due to the absolute and uniform convergence on compact subsets of the double series*

$$\sum_{\substack{\ell \geq 1 \\ j \geq 0}} p^{r(1-j)} w^\ell$$

1

*away from a pole.*

(2) *Recall that (see* [8, (2.1)]*)*

$$|\mathrm{GL}_n(\mathbb{F}_{p^r})| = p^{rn^2} f_{p^r}(n). \qquad (1.3)$$

*Therefore,* (1.1) *can be written as*

$$\sum_{n \geq 0} \frac{Q_{p^r}(n)}{|\mathrm{GL}_n(\mathbb{F}_{p^r})|} w^n = \prod_{\substack{\ell \geq 1 \\ j \geq 0}} \frac{1}{1 - p^{r(1-j)} w^\ell}.$$

(3) *By* [5, (1)] *and* (1.3), *we have*

$$Q_{p^r}(n) = |\mathrm{GL}_n(\mathbb{F}_{p^r})| \sum_{\lambda \vdash n} \frac{p^{r \sum_{k=1}^n b_k}}{\prod_{k=1}^n f_{p^r}(b_k)}, \qquad (1.4)$$

*where the sum runs over all partitions of $n$ and $\lambda$ has the frequencies $b_k$ given by $n = \sum_{k=1}^n b_k k$.*

A natural question is to approximate $Q_{p^r}(n)$, as $n \to \infty$. It was shown by Motzkin–Taussky [12] (see also [9, p. 71, Theorem 2]) that the space of $n \times n$ commuting matrices over an algebraically closed field has dimension $n^2 + n$. Hence, heuristically, $Q_{p^r}(n)$ should be proportional to $p^{r(n^2+n)}$ for $n$ sufficiently large. This observation was rigorously proved by Fulman and Guralnick [7, Theorem 2.6] as a byproduct of their new proof of the result of Feit and Fine. More precisely, Fulman and Guralnick established that

$$\lim_{n \to \infty} \frac{Q_{p^r}(n)}{p^{r(n^2+n)}} = \prod_{j \geq 1} \left(1 - p^{-rj}\right)^{-j}. \qquad (1.5)$$

In this paper, we prove more precise asymptotics for $Q_{p^r}(n)$. To start with, we write, for $w \in \mathbb{C}$ with $|w| < 1$,

$$\prod_{\substack{\ell \geq 1 \\ j \geq 0}} \frac{1}{1 - p^{r(1-j)} w^\ell} = P_{0,p^r}(w) F_{p^r}(w), \qquad (1.6)$$

where for $x > 0$

$$P_{m,x}(w) := \prod_{\substack{\ell \geq 1 \\ \ell \neq m}} \frac{1}{1 - xw^\ell}, \qquad F_{p^r}(w) := \prod_{\ell,j \geq 1} \frac{1}{1 - p^{r(1-j)} w^\ell}. \qquad (1.7)$$

We show the following asymptotic behavior of $Q_{p^r}(n)$, as $n \to \infty$. Throughout, we use the notation $\zeta_m := e^{\frac{2\pi i}{m}}$.

2

**Theorem 1.2.** *We have, as $n \to \infty$,*

$$Q_{p^r}(n) \sim p^{rn^2} f_{p^r}(n) \sum_{m \geq 1} C_{m,p^r}(n) p^{\frac{rn}{m}}, \qquad (1.8)$$

*where*

$$C_{m,p^r}(n) := \frac{1}{m} \sum_{j=0}^{m-1} P_{m,p^r}\left(\zeta_m^{-j} p^{-\frac{r}{m}}\right) F_{p^r}\left(\zeta_m^{-j} p^{-\frac{r}{m}}\right) \zeta_m^{nj}.$$

*In particular, we have, as $n \to \infty$,*

$$Q_{p^r}(n) = p^{r\left(n^2+n\right)} \prod_{j \geq 1} \left(1 - p^{-rj}\right)^{-j} + O\left(p^{rn^2 + \frac{rn}{2}}\right).$$

**Remark.** *The series in* (1.8) *does not converge, but should be viewed as an asymptotic expansion. Also note that $C_{m,p^r}(n)$ only depends on $n \pmod{m}$. In particular, $C_{1,p^r}(n)$ does not depend on $n$. Furthermore, we have*

$$|C_{m,p^r}(n)| \leq \max_{0 \leq j \leq m-1} \left|P_{m,p^r}\left(\zeta_m^{-j} p^{-\frac{r}{m}}\right) F_{p^r}\left(\zeta_m^{-j} p^{-\frac{r}{m}}\right)\right| = \left|P_{m,p^r}\left(p^{-\frac{r}{m}}\right) F_{p^r}\left(p^{-\frac{r}{m}}\right)\right|.$$

*As a result, when truncated at $m = N$, the error in* (1.8) *is $O(p^{\frac{rn}{N+1}})$.*

**Example.** Let $p = 2$ and $r = 1$. By (1.1), we have

$$(1 - 2w) \sum_{n \geq 0} \frac{Q_2(n)}{2^{n^2} f_2(n)} w^n = \prod_{\substack{\ell \geq 1 \\ j \geq 0 \\ (\ell,j) \neq (1,0)}} \frac{1}{1 - 2^{1-j} w^\ell}. \qquad (1.9)$$

While (1.1) has radius of convergence $\frac{1}{2}$, the right-hand side of (1.9) has radius of convergence $\frac{1}{\sqrt{2}}$. Hence, we obtain

$$C_{1,2}(1) = \lim_{n \to \infty} \frac{Q_2(n)}{2^{n^2+n} f_2(n)} = \prod_{j \geq 1} \frac{1}{(1 - 2^{-j})^{j+1}} = 34.738723457\ldots.$$

Further values are given by

$$\begin{aligned}
C_{2,2}(1) &= -11716.7651425569\ldots & C_{2,2}(2) &= -11716.3960075313\ldots \\
C_{3,2}(1) &= 7970793.64416118\ldots & C_{3,2}(2) &= 7970793.59033743\ldots \\
C_{3,2}(3) &= 7970793.67801128\ldots.
\end{aligned}$$

Looking at these first values one notes that they are very close to each other for $m$ fixed. A further investigation of their behavior would be interesting.

The paper is organized as follows. First we recall some basic definitions. Then we give a proof of Theorem 1.2, and generalize these ideas to another

3

Cohen–Lenstra series in Theorem 4.1. In the final section, we propose problems for further research.

## 2. Preliminaries

In the following, we recall some required results from complex analysis. We start with a well-known and simple observation that is still very useful for many aspects in the theory of $q$-series, recurrences, and combinatorics. For $\varrho \in \mathbb{R}^+$ let $B_\varrho(a) := \{z \in \mathbb{C} : |z - a| < \varrho\}$.

**Proposition 2.1.** *Let $a \in \mathbb{C}$, $\varrho \in \mathbb{R}^+$, and $f$ be a holomorphic function defined in a neighborhood of $a$. Then the following are equivalent:*

(1) *The function $f$ extends to an analytic function on $B_\varrho(a)$, but not on $B_{\varrho+\varepsilon}(a)$ for any $\varepsilon > 0$.*

(2) *The power series $\sum_{n\geq 0} \frac{f^{(n)}(a)}{n!}(z - a)^n$ has radius of convergence $\varrho$.*

For a sequence $a_m$ of non-zero complex numbers

$$F(w) := \prod_{m\geq 1}\left(1 - \frac{w}{a_m}\right). \tag{2.1}$$

**Proposition 2.2** (Wang, [13, Theorem 1]). *Let $a_m$ be a sequence of non-zero complex numbers such that $a_m \neq a_n$ for $m \neq n$ and such that $\sum_{m\geq 1}|a_m|^{-1}$ converges. Then the function $\frac{1}{F}$ is holomorphic in a neighborhood of $w = 0$ and we have there*

$$\frac{1}{F(w)} = -\sum_{k\geq 0}\sum_{m\geq 1}\frac{1}{F'(a_m)a_m^{k+1}}w^k.$$

## 3. Proof of Theorem 1.2

To prove Theorem 1.2, we require the following lemma.

**Lemma 3.1.** *Let $x > 1$, $n \in \mathbb{N}$, $0 < \varepsilon < x^{-\frac{1}{n}}$, and $f$ be a holomorphic function in some region $\mathcal{R}_{x,n,\varepsilon} := \{w \in \mathbb{C} \colon x^{-\frac{1}{n}} - \varepsilon < |w| < x^{-\frac{1}{n}} + \varepsilon\}$. Define*

$$g(w) := \frac{f(w)}{1 - xw^n}.$$

*Then*

$$g(w) - \frac{1}{n} \sum_{j=0}^{n-1} \frac{f\left(\zeta_n^{-j} x^{-\frac{1}{n}}\right)}{1 - \zeta_n^j x^{\frac{1}{n}} w}$$

*continues holomorphically to $\mathcal{R}_{x,n,\varepsilon}$.*

*Proof.* The only poles of $g$ in $\mathcal{R}_{x,n,\varepsilon}$ lie in $\{\zeta_n^{-j} x^{-\frac{1}{n}} \colon 0 \le j \le n-1\}$. These are simple. We compute, using l'Hospital,

$$\lim_{w \to \zeta_n^{-j} x^{-\frac{1}{n}}} \frac{\left(1 - \zeta_n^j x^{\frac{1}{n}} w\right) f(w)}{1 - xw^n} = \frac{1}{n} f\left(\zeta_n^{-j} x^{-\frac{1}{n}}\right).$$

The claim follows. $\qquad\square$

We have the following corollary to Lemma 3.1.

**Corollary 3.2.** *Let $f$ be a function holomorphic inside the unit disk and $N \in \mathbb{N}$. Then*

$$P_{0,p^r}(w)f(w) - \sum_{m=1}^{N} \frac{1}{m} \sum_{j=0}^{m-1} \frac{P_{m,p^r}\left(\zeta_m^{-j} p^{-\frac{r}{m}}\right) f\left(\zeta_n^{-j} p^{-\frac{r}{m}}\right)}{1 - \zeta_m^j p^{\frac{r}{m}} w}$$

*continues holomorphically to the region $\{w \in \mathbb{C} \colon |w| < p^{-\frac{r}{N+1}}\}$.*

*Proof.* By (1.7), we have, for $m \in \mathbb{N}$,

$$P_{0,p^r}(w)f(w) = \frac{P_{m,p^r}(w)f(w)}{1 - p^r w^m}.$$

Note that the numerator is holomorphic in the region

$$\left\{w \in \mathbb{C} \colon p^{-\frac{r}{m-1}} < |w| < p^{-\frac{r}{m+1}}\right\}$$

(where $p^{-\frac{r}{0}} := 0$). By applying Lemma 3.1 iteratively, the claim follows. $\qquad\square$

We are now ready to prove Theorem 1.2.

*Proof of Theorem 1.2.* Let $N \in \mathbb{N}$ be arbitrary. We abbreviate

$$c_{m,j} := P_{m,p^r}\left(\zeta_m^{-j}p^{-\frac{r}{m}}\right) F_{p^r}\left(\zeta_m^{-j}p^{-\frac{r}{m}}\right).$$

By (1.1), (1.6), and Corollary 3.2 we conclude that

$$\sum_{n \geq 0} \frac{Q_{p^r}(n)}{p^{rn^2} f_{p^r}(n)} w^n - \sum_{m=1}^{N} \frac{1}{m} \sum_{j=0}^{m-1} \frac{c_{m,j}}{1 - \zeta_m^j p^{\frac{r}{m}} w}$$

continues holomorphically to $\{w \in \mathbb{C} : |w| < p^{-\frac{r}{N+1}}\}$. We can then write

$$\sum_{m=1}^{N} \frac{1}{m} \sum_{j=0}^{m-1} \frac{c_{m,j}}{1 - \zeta_m^j p^{\frac{r}{m}} w} = \sum_{n \geq 0} \sum_{m=1}^{N} \frac{p^{\frac{rn}{m}} w^n}{m} \sum_{j=0}^{m-1} c_{m,j} \zeta_m^{jn}.$$

Hence we have, for $\varepsilon > 0$,

$$\frac{Q_{p^r}(n)}{p^{rn^2} f_{p^r}(n)} - \sum_{m=1}^{N} \frac{1}{m} \sum_{j=0}^{m-1} c_{m,j} \zeta_m^{jn} p^{\frac{rn}{m}} = O_\varepsilon\left(p^{\left(\frac{r}{N+1}+\varepsilon\right)n}\right) \quad (n \to \infty).$$

Equivalently, we have, as $n \to \infty$,

$$Q_{p^r}(n) = p^{rn^2} f_{p^r}(n) \sum_{m=1}^{N} \frac{1}{m} \sum_{j=0}^{m-1} c_{m,j} \zeta_m^{jn} p^{\frac{rn}{m}} + O_\varepsilon\left(p^{rn^2 + \left(\frac{r}{N+1}+\varepsilon\right)n}\right).$$

The result follows after a straightforward calculation. $\qquad\square$

Finally, we give an upper bound for the coefficients in the asymptotic expansion in Theorem 1.2.

**Proposition 3.3.** *We have, for $\varepsilon > 0$, uniformly in $m$, $p^r$, and $n$*

$$C_{m,p^r}(n) \ll_\varepsilon \exp\left(\frac{\zeta(3) + \varepsilon}{\left(1 - p^{-\frac{r}{m}}\right)^2}\right),$$

*where $\zeta$ denotes the Riemann zeta function.*

*Proof.* By (1.7) we have

$$\left|P_{m,p^r}\left(p^{-\frac{r}{m}}\right) F_{p^r}\left(p^{-\frac{r}{m}}\right)\right| = \prod_{\substack{\ell \geq 1 \\ \ell \neq m}} \left|\frac{1}{1 - p^{r-\frac{r\ell}{m}}}\right| \prod_{\ell,j \geq 1} \left|\frac{1}{1 - p^{r(1-j)} p^{-\frac{r\ell}{m}}}\right|.$$

We estimate both products separately. First, we have

$$\prod_{\substack{\ell \geq 1 \\ \ell \neq m}} \frac{1}{1 - p^{r-\frac{r\ell}{m}}} = (-1)^{m+1} p^{-\frac{rm(m+1)}{2m}} P_{0,1}\left(p^{-\frac{r}{m}}\right) \prod_{\ell=1}^{m-1} \frac{1}{1 - p^{-\frac{r\ell}{m}}}$$

6

$$\ll p^{-\frac{r(m-1)}{2}} P_{0,1}^2 \left( p^{-\frac{r}{m}} \right).$$

For the second product, we find

$$\prod_{\ell,j\geq 1} \frac{1}{1-p^{r(1-j)}p^{-\frac{r\ell}{m}}} = \prod_{\substack{j\geq 0 \\ \ell\geq 1}} \frac{1}{1-p^{-\frac{r}{m}(mj+\ell)}} \ll \prod_{\substack{j\geq 0 \\ \ell\geq 1}} \frac{1}{1-p^{-\frac{r}{m}(j+\ell)}} = \prod_{j\geq 1} \left(1-p^{-\frac{rj}{m}}\right)^{-j}.$$

Applying [2, Lemma 3.2], we have, as $w \to 1^-$,

$$P_{0,1}^2(w) \ll_\varepsilon e^{\frac{\zeta(2)+\varepsilon}{|w-1|}}, \qquad \prod_{j\geq 1} \frac{1}{(1-w^j)^j} \ll_\varepsilon e^{\frac{\zeta(3)+\varepsilon}{|w-1|^2}}$$

for all $\varepsilon > 0$. As a result, for all $\varepsilon > 0$, we find the claimed bound. $\qquad\square$

## 4. Cohen–Lenstra Series and the Nilpotent Classes

Since its discovery, Feit and Fine's generating function (1.1) has been re-proved by other means such as the motivic Donaldson–Thomas theory [3]. More recently, Huang [10] placed such counting problems into the framework of Cohen–Lenstra series [4]. Simply speaking, Huang considered *(generalized) Cohen–Lenstra series*, which are defined as

$$Z_R(w) = Z_{R/\mathbb{F}_{p^r}}(w) := \sum_M \frac{1}{|\operatorname{Aut}(M)|} w^{\dim_{\mathbb{F}_{p^r}}(M)},$$

where $M$ runs over all isomorphism classes of finite $R$-modules and $R$ is a commutative ring containing the finite field $\mathbb{F}_{p^r}$ such that the above series is well-defined[1]. A more general setting was later provided by Huang in [11]. Note that, by (1.3), the relation (1.1) corresponds to the Cohen–Lenstra series (see also item 3 in Huang's list in [10, p. 27]):

$$Z_{\mathbb{F}_{p^r}[u,v]}(w) = \sum_{n\geq 0} \frac{\left|\{(A,B) \in \operatorname{Mat}_n(\mathbb{F}_{p^r})^2 : AB = BA\}\right|}{|\operatorname{GL}_n(\mathbb{F}_{p^r})|} w^n = \prod_{\substack{\ell\geq 1 \\ j\geq 0}} \frac{1}{1-p^{r(1-j)}w^\ell},$$

where, as usual, $\operatorname{Mat}_n(\mathbb{F}_{p^r})$ denotes the set of $n \times n$ matrices over $\mathbb{F}_{p^r}$. In this setting, we may generalize $R$ to count commuting varieties (i.e., subvarieties of $XY - YX = 0$) over a finite field; see [10, p. 40, Proposition 4.3]. However, these Cohen–Lenstra series in general do not admit a product form and mostly behave multisum-like, see [10, (1.14) and (1.15)]. Therefore, our method of proof used for Theorem 1.2 does not apply. To work out new methods to deal with these products would be worthwhile.

---

[1]The original Cohen–Lenstra series in [4] are defined for $R$ a Dedekind domain over $\mathbb{F}_{p^r}$.

In the meantime, we can restrict the enumeration of commuting square matrix pairs to nilpotent classes. Let

$$\mathrm{Nilp}_n(\mathbb{F}_{p^r}) := \big\{A \in \mathrm{Mat}_n(\mathbb{F}_{p^r}) : A \text{ is nilpotent}\big\}.$$

In the simplest cases, due to Fine and Herstein [6] and Fulman and Guralnick [7], we still have product-like generating functions similar to (1.1). In terms of the Cohen–Lenstra series, the two cases may be written as follows:

- $R = \mathbb{F}_{p^r}[[u]]$ – Fine–Herstein [6, p. 499, Theorem 1] (see also item 2 on the list of Huang [10, p. 27]):

$$Z_{\mathbb{F}_{p^r}[[u]]}(w) = \sum_{n\geq 0} \frac{|\mathrm{Nilp}_n(\mathbb{F}_{p^r})|}{|\mathrm{GL}_n(\mathbb{F}_{p^r})|} w^n = \prod_{j\geq 1} \frac{1}{1 - p^{-rj}w}, \qquad (4.1)$$

- $R = \mathbb{F}_{p^r}[[u,v]]$ – Fulman–Guralnick [7, p. 301, Theorem 2.9] (see also item 4 on the list of Huang [10, p. 27]):

$$Z_{\mathbb{F}_{p^r}[[u,v]]}(w) = \sum_{n\geq 0} \frac{|\{(A,B)\in\mathrm{Nilp}_n(\mathbb{F}_{p^r})^2 : AB=BA\}|}{|\mathrm{GL}_n(\mathbb{F}_{p^r})|} w^n$$

$$= \prod_{\substack{\ell\geq 1 \\ j\geq 0}} \frac{1}{1 - p^{-r(1+j)}w^\ell}. \qquad (4.2)$$

The structure of (4.1) is particularly simple, which allows us to use Proposition 2.2 to derive not only an asymptotic expression but even a closed-form series representation for its coefficients. For $w \in \mathbb{C}$, we define

$$Z_{m,\mathbb{F}_{p^r}[[u]]}(w) := \prod_{\substack{j\geq 1 \\ j\neq m}} \frac{1}{1 - p^{-rj}w}.$$

**Theorem 4.1.** *Let $p$ be a prime and $r \in \mathbb{N}$. Then we have, for $n \in \mathbb{N}_0$,*

$$\frac{|\mathrm{Nilp}_n(\mathbb{F}_{p^r})|}{|\mathrm{GL}_n(\mathbb{F}_{p^r})|} = \sum_{m\geq 1} \frac{Z_{m,\mathbb{F}_{p^r}[[u]]}(p^{rm})}{p^{rnm}}.$$

*Proof.* We apply Proposition 2.2 with $a_j := p^{rj}$. As $p^r > 1$, the series $\sum_{m\geq 1} p^{-rm}$ converges and $a_m \neq a_n$ for all $m \neq n$. By the product rule (see (2.1) for the definition of $F$)

$$F'(p^{rm}) = -\frac{1}{p^{rm} Z_{m,\mathbb{F}_{p^r}[[u]]}(p^{rm})}.$$

Using this, we obtain the theorem. $\qquad\square$

**Example.** The series in Theorem 4.1 is actually convergent. We choose $p = 2$ and $r = 1$ in this example. And let

$$Z_{m,\mathbb{F}_2[[u]],N}(w) := \prod_{\substack{N \geq j \geq 1 \\ j \neq m}} \frac{1}{1 - 2^{-j}w}.$$

We have

$$Z_{\mathbb{F}_2[[u]]}(w) = \prod_{j \geq 1} \frac{1}{1 - 2^{-j}w} = 1 + w + \frac{2}{3}w^2 + \frac{4}{11}w^3 + \frac{64}{315}w^4 + \cdots .$$

On the other hand, for $N = 100$, one finds numerically

$$\sum_{1 \leq m \leq 10} Z_{m,\mathbb{F}_2[[u]],100}(2^m) = 0.999999999999999667\ldots,$$

$$\sum_{1 \leq m \leq 10} \frac{Z_{m,\mathbb{F}_2[[u]],100}(2^m)}{2^m} = 0.9999999999999999998\ldots,$$

$$\sum_{1 \leq m \leq 10} \frac{Z_{m,\mathbb{F}_2[[u]],100}(2^m)}{2^{2m}} = 0.6666666666666666666\ldots,$$

$$\sum_{1 \leq m \leq 10} \frac{Z_{m,\mathbb{F}_2[[u]],100}(2^m)}{2^{3m}} = 0.3809523809523809523\ldots,$$

$$\sum_{1 \leq m \leq 10} \frac{Z_{m,\mathbb{F}_2[[u]],100}(2^m)}{2^{4m}} = 0.2031746031746031746\ldots.$$

These sums represent the cases $0 \leq n \leq 4$. Note that also the degenerate case $n = 0$ is included.

## 5. Questions for future research

We propose the following research questions.

(1) Is there any recursive formula for the $Q_{p^r}(n)$ (or quotients of it by normalization factors) that is analogous to the divisor recurrence formula/the Euler Pentagonal Theorem [1, (1.3.1)]:

$$\prod_{n \geq 1}(1 - q^n) = \sum_{n \in \mathbb{Z}}(-1)^n q^{\frac{n(3n-1)}{2}}?$$

In the second scenario there is no exact identity but rather an error term smaller than all exponential functions in the main series. Is there a combinatorial interpretation for these recurrences and the error term?

(2) Can we improve the bounds for $C_{m,p^r}(n)$? What can we say about these coefficients for fixed $m$?

(3) How could the methods in this paper be extended to deal with multisum-like products as (4.2)?

## References

[1] G. Andrews, *The theory of partitions*, Cambridge University Press, Cambridge, 1998.

[2] W. Bridges, B. Brindle, K. Bringmann, and J. Franke, *Asymptotic expansion for partitions generated by infinite products*, Math. Ann. **390** (2024), 2593–2632.

[3] J. Bryan and A. Morrison, *Motivic classes of commuting varieties via power structures*, J. Algebraic Geom. **24** (1) (2015), 183–199.

[4] H. Cohen and H. Lenstra, *Heuristics on class groups of number fields*, in: Number theory, Noordwijkerhout 1983, 33–62, Springer, Berlin, 1984.

[5] W. Feit and N. Fine, *Pairs of commuting matrices over a finite field*, Duke Math. J. **27** (1) (1960), 91–94.

[6] N. Fine and I. Herstein, *The probability that a matrix be nilpotent*, Illinois J. Math. **2** (1958), 499–504.

[7] J. Fulman and R. Guralnick, *Enumeration of commuting pairs in Lie algebras over finite fields.* Annals of Combinatorics **22** (2018), 295–316.

[8] J. Fulman, R. Guralnick, and D. Stanton, *Asymptotics of the number of involutions in finite classical groups.* J. Group Theory **20** (2017), 871–902.

[9] R. Guralnick, *A note on commuting pairs of matrices*, Linear and multilinear algebra **31** (1-4) (1992), 71–75.

[10] Y. Huang, *Mutually annihilating matrices, and a Cohen–Lenstra series for the nodal singularity*, J. Algebra **619** (2023), 26–50.

[11] Y. Huang, *Commuting matrices via commuting endomorphisms*, arXiv:2404.19483.

[12] T. Motzkin and O. Taussky, *Pairs of matrices with property L. II.*, Trans. Amer. Math. Soc. **80** (1955), 387–401.

[13] X. Wang, *The summation of infinite partial fraction decomposition I: some formulae related to the Hurwitz zeta function*, arXiv:2102.04115.

University of Cologne, Department of Mathematics and Computer Science, Weyertal 86-90, 50931 Cologne, Germany

*Email address*: kbringma@uni-koeln.de

*Email address*: jfrank12@uni-koeln.de

*Email address*: bheim@uni-koeln.de

Fakultät für Mathematik, Universität Wien, Oskar-Morgenstern-Platz 1, Wien 1090, Austria

*Email address*: xiaohangc92@univie.ac.at, chenxiaohang92@gmail.com